Welcome All!

- All admin stuff is on canvas
  - see the video I posted

Yiddish word of the day

"oysgetseykhnt" = אויסגעצײכנט

"excellent" =

Yiddish curse / blessing / expression of the day

"likht vi esn bagel" = לײַכט ווי עסן בײגל

"as easy as eating bagels" =

So why Linear Algebra?

- This is the question we will explore in the class
- Moreover, there is the final project you all will complete about this

Humble Origins

1) How to solve $ax = b$ $\quad x = \frac{b}{a} \; (a \neq 0)$

2) $2x - y = 2$
   $x + y = 4$
   $\overset{?}{\Rightarrow} x = 4 - y \overset{\text{plug in}}{\Rightarrow} 2(4-y) - y = 2$

3) $x + y + z = 2$
   $-x - 2y + 3z = 4$
   $x + 11y - z = 6$
   $\overset{?}{\Rightarrow}$ This sucks.

- Turns out, a lot of questions in "nature" can be expressed as a system of equations.

- Want some better way of solving these.

Some notation

- We'll use $x_1, x_2, x_3, \ldots$ as variables.

First thing to Note

- If a system of equations looks like this

$$
\begin{aligned}
x_1 + 3x_2 - 4x_3 &= 1 \\
2x_2 - x_3 &= 0 \\
x_3 &= 4
\end{aligned}
$$

we can **back substitute** the variables.

Def: 1) The leading variable in an equation is the first nonzero variable.

2) A free variable in an equation are the variables "to the right" of the leading variables.

3) A linear system of equations is in Echelon Form (EF) if the indices of the leading variable are increasing as you move down.

ex 1)
$$\begin{aligned} 3x_1 - 4x_2 + 6x_3 &= 8 \\ x_2 - x_3 &= 4 \\ x_2 + 10x_3 &= 6 \end{aligned}$$

In / Not in EF?

2) $x_1 - 4x_2 - x_3 + x_4 = 1$

$-10x_3 - 3x_4 = 2$

$x_4 = 1$

In / Not in EF?

3) $5x_1 - 2x_2 + 6x_3 = 6$

$x_2 - x_3 = 0$

$x_1 = 4$

In / Not in EF?

Goal Want to convert a "general" system like this

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix} \quad (A)$$

into echelon Form!

There are 3 elementary row operations that we can "do" to our system to put it into Echelon Form

1) Swap rows

2) Multiply a row by a #

3) Add a multiple of a row to another row.

ex 1) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + x_2 + 3x_3 &= 2 \\ -x_1 + 3x_2 + 3x_3 &= 5 \end{aligned}$$

Goal: Put this into Echelon Form

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + x_2 + 3x_3 &= 2 \\ -x_1 + 3x_2 + 3x_3 &= 5 \end{aligned} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 5x_2 + x_3 &= 6 \\ -x_1 + 3x_2 + 3x_3 &= 5 \end{aligned} \xrightarrow{R_3 \to R_1 + R_3} \begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 5x_2 + x_3 &= 6 \\ 5x_2 + 2x_3 &= 7 \end{aligned}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \underbrace{\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 5x_2 + x_3 &= 6 \\ x_3 &= 1 \end{aligned}}_{\text{in EF}} \implies \begin{aligned} x_1 + 2 - 1 &= 2 \\ x_2 &= 1 \\ x_3 &= 1 \end{aligned} \implies \boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \end{aligned}}$$

ex2) $x_1 + x_2 + x_3 = 1$ $\quad R_2 \to R_2 - R_1$ $\quad x_1 + x_2 + x_3 = 1$

$x_1 + 2x_2 + 3x_3 = 2$ $\quad\longrightarrow\quad$ $x_2 + 2x_3 = 1$

$2x_1 + x_2 + 0x_3 = 1$ $\quad R_3 \to R_3 - 2R_1$ $\quad -x_2 - 2x_3 = -1$

$R_3 \to R_3 + R_2$ $\longrightarrow$ $x_1 + x_2 + x_3 = 1$

$x_2 + 2x_3 = 1$

$0x_2 + 0x_3 = 0$

Note, in this case x_3 is a free-variable

It can be any # we want,

(ie, it is not a leading variable in any row)

In these 2 examples

1) No "zero rows" and no free variables = unique solution

2) A zero row and a free variable = infinite #

ex 3:

$$x_1 - 3x_2 + 2x_3 = 0$$
$$2x_1 - 5x_2 + 4x_3 = 1$$
$$x_1 - 4x_2 + 2x_3 = -2$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$x_1 - 3x_2 + 2x_3 = 0$$
$$x_2 + 0x_3 = 1$$
$$-x_2 + 0x_3 = -2$$

$R_3 \to R_2 + R_3$

$$x_1 - 3x_2 + 2x_3 = 0$$
$$x_2 = 1$$
$$0x_2 + 0x_3 = -1$$

zero row = non-zero #

No solution!!

Note: If your system looks like

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

then there is always a

We call a system like this $\boxed{\text{homogeneous system}}$

In Summary

Start with Linear System
- Put into Echelon Form

1 unique solution if no free variables

∞-many solutions if there is free variable

no solution if zero row = nonzero #

Section 1.2 - Matrices

In these linear systems really care about the coefficients in front of the variables

$$a_{11}x_1 + \cdots + a_{1m}x_m = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

$$\longrightarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ & & & & b_2 \\ & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

this matrix is called the coefficient matrix of the linear system.

ex) Write coefficient matrix for

$$3x_1 - x_2 + x_3 = 4$$
$$-2x_1 - x_2 + 2x_3 = 1$$
$$x_1 - 4x_2 + 6x_3 = 3$$

$$\longrightarrow \left(\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ -2 & -1 & 2 & 1 \\ 1 & -4 & 6 & 3 \end{array}\right)$$

Def: A matrix is in Echelon Form (EF) if

1) The first nonzero entry of each row is to the right of the above row
2) Any zero rows are at the bottom

3) The first nonzero entry in a row is 1

ex) i) $\left(\begin{array}{ccc|c} 3 & -3 & 6 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 1 & 10 & 6 \end{array}\right)$ In / Not in EF?

ii) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$ In / Not in EF?

Def: The positions of the leading terms in a row are called pivot positions

The collumns of the matrix that have a pivot positions are called pivot collumns

Q: How to put a matrix in EF? A: Same 3 row operations

ex) $\left(\begin{array}{ccc|c} 3 & -3 & 6 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 1 & 10 & 6 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 3 & -3 & 6 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 11 & 2 \end{array}\right)$

$\Rightarrow x_3 = \frac{2}{11}$ (plug into row 2) $x_2 - \frac{2}{11} = 4$ ....

ex)
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

Is there an issue with this coefficient matrix?

Zero row = non-zero #

What You Can Do Now

- Write the coefficient matrix for Linear System
- Put Matrix into Echelon Form
- Tell if system has a solution

Where we go From Here

1) Talk about $\mathbb{R}^n$

2) Rephrase our questions we asked today in terms of "vectors" in $\mathbb{R}^n$ and "spanning"